

# Physical accessible transformations on a finite number of quantum states

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We consider to treat the usual probabilistic cloning, state separation, unambiguous state discrimination, *etc* in a uniform framework. All these transformations can be regarded as special examples of generalized completely positive trace non-increasing maps on a finite number of input states. From the system-ancilla model we construct the corresponding unitary implementation of pure  $\rightarrow$  pure, pure  $\rightarrow$  mixed, mixed  $\rightarrow$  pure, and mixed  $\rightarrow$  mixed states transformations in the whole system and obtain the necessary and sufficient conditions on the existence of the desired maps. We expect our work will be helpful to explore what we can do on a finite set of input states.

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## I. INTRODUCTION

Recently, rapid progress has been made on quantum information science. Superposition and entanglement play the central roles and make it quite different from its classical correspondence. Benefited from these novel properties many interesting applications have been proposed in the past years, such as dense coding [1], quantum teleportation [2], quantum cryptography [3], *etc*. On the other hand, quantum superposition also puts many constraints on physical realizable manipulation on quantum states. For example, we cannot copy (or delete) an unknown qubit. This statement is known as the no-cloning (no-deleting) principle and constitutes one of the most significant differences between classical and quantum information. Hence to get to know what we can and cannot do for a given set of quantum states becomes an important and interesting problem.

Many efforts have been devoted into this question. One example is the universal quantum operation (universal quantum state cloning [4], universal state estimation [5], *etc*). In the universal quantum operation the input states compose of the whole Hilbert space, and usually such a transformation can be carried out determinately with the fidelity be independent of the input states. In the most practical cases, people often get to know some specific information about the input states set and particularly the input usually contains only a finite number of quantum states. Therefore the problem we focus on in this paper is: given a finite set of input states  $\{\rho_1, \rho_2, \dots, \rho_n\}$  ( $\mathcal{H}_1$ ) and the corresponding output states  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  ( $\mathcal{H}_2$ ), we consider whether there exists a physical accessible operation  $\xi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  to implement such kind of transformation. Here  $\rho_i$  ( $\sigma_i$ ) denotes the input (output) state density matrix, and it can be pure or mixed. We also require the operation

should be accurate and probabilistic. Many interesting things can be enclosed in the framework, such as probabilistic cloning [6], state separation [7], unambiguous states discrimination [8], *etc*, and actually they are only some particular cases of general completely positive (CP) maps between two finite sets of input states.

Usually any physical evolution can be described by a CP trace non-increasing map. There are many other equivalent ways to illustrate such transformation. For example, it can be represented in an operator sum form  $\xi(\rho) = \sum_k M_k \rho M_k^\dagger$  where  $M_k$  are the Kraus operators and satisfy  $\sum_k M_k^\dagger M_k \leq I$  [9]. Also it can be implemented by employing a unitary transformation on the extended Hilbert space. The corresponding mathematical description can be written as  $\xi(\rho) = \text{Tr}_{E'}[U \rho \otimes \rho_E U^\dagger I \otimes P_{E'}]$  [10], where  $\rho \in \mathcal{H}_1$ ,  $\mathcal{H}_1 \otimes \mathcal{H}_E = \mathcal{H}_2 \otimes \mathcal{H}_{E'}$ ,  $\rho_E$  is the initial state of the ancilla,  $I$  denote the identity operator in output Hilbert space  $\mathcal{H}_2$ , and  $P_{E'}$  is a projector in  $\mathcal{H}_{E'}$ . In our consideration we choose the second kind of description since it is easy understandable and has been widely used in many related works [6].

The paper is organized as follows. In section II, under some simple assumptions we consider the sufficient and necessary conditions about the existence of the CP map  $\xi$ . We divide the whole question into four parts, that is pure  $\rightarrow$  pure, pure  $\rightarrow$  mixed, mixed  $\rightarrow$  pure, and mixed  $\rightarrow$  mixed states cases. From the system-ancilla model we obtain the sufficient and necessary conditions on the existence of physical accessible operations. We also examine the consequences of these conditions under some special examples, such as probabilistic cloning of linear independent quantum states, state separation, unambiguous state discrimination, *etc*. Compared with the method used in already existing results, we think our results are more apparent and easier understandable. In section III, we consider the influence of the auxiliary system to whole problem, which has been neglected in some related works [6], and propose a generalized probabilistic cloning machine. We also give a brief discussion of its relationship to standard semidefinite programming. We conclude our remarks in section VI.

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## II. CP TRACE NON-INCREASING MAP ON A FINITE SET OF INPUT STATES

In this section, we try to find the sufficient and necessary conditions on the existence of the map  $\xi$ . Generally this question is quite complicated. To simplify our consideration, we assume that the initial state of the auxiliary system is a pure state and divide the whole problem into four parts. We think this will be helpful to understand the main results of this paper.

### A. CP trace non-increasing map between two pure states sets

Let us begin with the pure-state to pure-state case, where the input and output states are both pure and can be represented as  $\xi : \{|\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_n\rangle\} \rightarrow \{|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle\}$ . This kind of transformation has been studied in many related works, for example, probabilistic cloning and state separation, where the input states are required to be linear independent. However, for a general pure-state to pure-state transformation, this requirement can be loosen. Additionally if there exists a quantum operation  $\xi$  to complete such kind of map, one can always construct the following unitary transformation

$$U|\varphi_i\rangle_1|0\rangle_E = \sqrt{\eta_i}|\phi_i\rangle_2|\alpha_i\rangle_a|P_0\rangle_p + |\tilde{\beta}_i\rangle_{2ap}, \{1 \leq i \leq n\}, (1)$$

where  $\mathcal{H}_{E'} = \mathcal{H}_a \otimes \mathcal{H}_p$ ,  $|P_0\rangle$  is the state of the probe system satisfying  $\langle P_0|\beta_i\rangle = 0$ , and  $\eta_i$  is the success probability. Here and in the following, we use the tilde  $\sim$  to denote a nonnormalized state vector and omit the subscriptions for simplicity. Since any unitary transformation preserves inner-product, we can easily obtain  $X = \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A + B$  with  $X_{ij} = \langle\varphi_i|\varphi_j\rangle$ ,  $\sqrt{\Gamma} = \text{diag}\{\sqrt{\eta_1}, \sqrt{\eta_2}, \dots\}$ ,  $Y_{ij} = \langle\phi_i|\phi_j\rangle$ ,  $A_{ij} = \langle\alpha_i|\alpha_j\rangle$ , and  $B_{ij} = \langle\tilde{\beta}_i|\tilde{\beta}_j\rangle$ . It is not difficult to demonstrate that the two matrices  $A$  and  $B$  are both positive semidefinite. Thus we obtain

$$X - \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A = B \geq 0, \quad (2)$$

where “ $\circ$ ” denotes the Hadamard (or Schur) matrix product.

On the other hand, if one can find an efficiency matrix  $\Gamma$  and a positive semidefinite matrix  $A$  satisfying Eq. (2), we can construct a unitary transformation to realize Eq. (1). Since  $X - \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A$  is Hermitian and positive semidefinite, it can be diagonalized by unitary transformation

$$V[X - \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A]V^\dagger = \text{diag}\{b_1, b_2, \dots, b_n\}, \\ \forall m, b_m \geq 0. \quad (3)$$

By choosing  $n$  orthogonal states  $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$  satisfying  $\langle P_0|k\rangle = 0$  appropriately and setting

$|\tilde{\beta}_i\rangle = \sum_{k=1}^m V_{ki}\sqrt{b_k}|k\rangle$ , we can check that  $\langle\tilde{\beta}_i|\tilde{\beta}_j\rangle = \sum_k V_{ik}^\dagger b_k V_{kj} = [X - \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A]_{ij}$ . Hence with standard Gram-Schmidt process the desired unitary map can be constructed to satisfy Eq. (1).

We conclude the above discussion by the following lemma.

**Lemma 1.** *For any given state secretly chosen from the set  $\{|\varphi_1\rangle, \dots, |\varphi_n\rangle\}$ , it can be transformed to the corresponding output state in  $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$  with the success probability matrix  $\Gamma = \text{diag}\{\eta_1, \dots, \eta_n\}$  if and only if there exists a positive semidefinite matrix  $A$  with the diagonal elements  $A_{ii} = 1$  such that  $X - \sqrt{\Gamma}Y\sqrt{\Gamma^\dagger} \circ A \geq 0$ . If the input states are chosen with prior probability  $\{p_1, \dots, p_n\}$  and  $\sum_i p_i = 1$ , then the whole success probability will be  $P = \sum_i p_i \eta_i$ .*

This lemma characterizes the general property of physical accessible maps between pure states. Usually it does not give any constraints on the input states, that means  $\{|\varphi_1\rangle, \dots, |\varphi_n\rangle\}$  can be linear dependent or linear independent. However, when the output states are linear independent, the matrix  $Y$  will be full-rank  $\text{rank}(Y) = \text{rank}(Y \circ A) = n$ . In order to implement the CP map with non-zero efficiency  $\eta_k > 0$ , the matrix  $X$  also must be full-rank, this leads to the linear independence of the input states. Many interesting questions can be enclosed into this case, for example, probabilistic copy and state discrimination of pure states. Since the output states are linear independent, the transformation become possible only when the input states are also linear independent.

As a special example, in the following we concentrate on deterministic transformation between pure states. This problem has been considered by Chefles [11] using different method, while in our framework, it becomes more apparent and easier understandable. Since  $\Gamma$  is identity matrix, according to our lemma, we obtain

$$X - Y \circ A = 0 \iff \langle\varphi_i|\varphi_j\rangle = \langle\phi_i|\phi_j\rangle\langle\alpha_i|\alpha_j\rangle. \quad (4)$$

Eq. (4) tells us that the modules of the overlap of the initial states should be less than their final correspondence. If one can find a positive semidefinite matrix  $A$  with  $A_{ii} = 1$  satisfying  $X = Y \circ A$ , a deterministic transformation between the two state sets can be realized. Additionally, if  $\langle\phi_i|\phi_j\rangle \neq 0$ , the matrix  $A$  can be easily constructed as  $A_{ij} = \langle\varphi_i|\varphi_j\rangle/\langle\phi_i|\phi_j\rangle$ . Therefore by judging whether  $A$  is positive semidefinite or not, we can get to know the existence of a deterministic transformation.

In the case of unambiguous discrimination between pure states, the output states are orthogonal to each other. This leads to a simplified version of Eq.(2)

$$X - \Gamma \geq 0. \quad (5)$$

We then conclude that  $N$  pure states can be unambiguously discriminated if and only if there exists an efficiency matrix  $\Gamma$  such that  $X - \Gamma$  is positive semidefinite, which has been discussed in many related works [6, 7].

### B. CP trace non-increasing map between pure states and mixed states

Now we continue our consideration by assuming the output states to be mixed  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ . This indicates that we should find the map  $\xi(|\varphi_i\rangle\langle\varphi_i|) = \sigma_i$  with  $\sigma_i$  be mixed state. Since the initial state of the auxiliary system is a fixed pure state, from the system-ancilla model we can obtain that the transformation can always be written as follows

$$U|\varphi_i\rangle_1|0\rangle_E = \sqrt{\eta_i}|\phi_i\rangle_{2a}|P_0\rangle_p + |\tilde{\beta}_i\rangle_{2ap}. \quad (6)$$

The output states can be obtained by tracing out the subsystem  $a$  after a measurement on the probe, *i.e.*  $\sigma_i = \text{Tr}_a(|\phi_i\rangle\langle\phi_i|)$ . Therefore  $|\phi_i\rangle_{2a}$  is nothing but a purification of the output state  $\sigma_i$ . Thus by introducing the purification of the output mixed state, the question we considered now can be reduced to the former case where both of the input and output states are pure. This leads to the following lemma.

**Lemma 2.** *For any given state secretly chosen from the set  $\{|\varphi_1\rangle, \dots, |\varphi_n\rangle\}$ , it can be transformed to the corresponding mixed output state in  $\{\sigma_1, \dots, \sigma_n\}$  with the success probability matrix  $\Gamma = \text{diag}\{\eta_1, \dots, \eta_n\}$  if and only if  $X - \sqrt{\Gamma}Y\sqrt{\Gamma}^\dagger \geq 0$ , where  $Y_{ij} = \langle\phi_i|\phi_j\rangle$  with  $|\phi_i\rangle$  be the purification of the output state  $\sigma_i$ . Additionally if the input states are chosen with prior probability  $\{p_1, \dots, p_n\}$  and  $\sum_i p_i = 1$ , then the whole success probability will be  $P = \sum_i p_i \eta_i$ .*

Since for any given mixed state, there are infinite types of purified states. What's more, the purified states are usually entangled states. Therefore the question we considered now becomes quite different from the former case. For example, in the case of deterministic transformation, one has  $X = Y$ , which means there exists a set of purified states  $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$  such that the corresponding inner-product equals to the case of input states. This can be regarded as a generalized version of pure-state to pure-state case, since  $|\phi_i\rangle|\alpha_i\rangle$  is also a purification of  $|\phi_i\rangle$ . However one cannot obtain a corresponding positive semidefinite matrix  $A$  easily because a purified state of  $\sigma_i$  is usually an entangled state.

The requirement of the input states to be linear independent is also different now. Generally the linear independency of  $\sigma_i$  cannot assert that the corresponding purifications are also linear independent. But if the support of any output state is not contained in the combinations of the supports of the rest mixed states, that is  $\text{supp}(\sigma_i) \not\subseteq \bigoplus_{k \neq i}^n \text{supp}(\sigma_k)$ , then any purified states  $\{|\phi_i\rangle\}$  are linear independent, and  $Y$  is an invertible positive definite matrix. From the lemma, we obtain that the input states must be linear independent.

In the case of two input states ( $n = 2$ ), the condition in lemma 2 can be written as

$$\begin{pmatrix} 1 - \eta_1 & \langle\varphi_1|\varphi_2\rangle - \sqrt{\eta_1\eta_2}\langle\phi_1|\phi_2\rangle \\ \langle\varphi_2|\varphi_1\rangle - \sqrt{\eta_1\eta_2}\langle\phi_2|\phi_1\rangle & 1 - \eta_2 \end{pmatrix} \geq 0.$$

This implies

$$\sqrt{(1 - \eta_1)(1 - \eta_2)} \geq |\langle\varphi_1|\varphi_2\rangle| - \sqrt{\eta_1\eta_2}|\langle\phi_1|\phi_2\rangle|. \quad (7)$$

A simple algebra leads us to the following bound of the whole success probability

$$\begin{aligned} P = p_1\eta_1 + p_2\eta_2 &\leq \frac{1 - 2\sqrt{p_1p_2}|\langle\varphi_1|\varphi_2\rangle|}{1 - |\langle\phi_1|\phi_2\rangle|} \\ &\leq \frac{1 - 2\sqrt{p_1p_2}|\langle\varphi_1|\varphi_2\rangle|}{1 - F(\sigma_1, \sigma_2)}. \end{aligned} \quad (8)$$

Here  $F(\sigma_1, \sigma_2)$  is the fidelity of the two output state which is defined as  $F(\sigma_1, \sigma_2) = \text{Tr}(\sqrt{\sqrt{\sigma_1}\sigma_2\sqrt{\sigma_1}})$ . The last step of the Eq. (8) comes from the well-known result that  $F(\sigma_1, \sigma_2) = \max_{\{|\phi_1\rangle, |\phi_2\rangle\}} |\langle\phi_1|\phi_2\rangle|$ , where the maximization runs over all purified states [12, 13].

### C. CP trace non-increasing map between mixed states and pure states

In the above discussion, we assume that the input states are all pure states. Here and in the following, we consider what happens if the input set is composed of mixed states. This problem becomes a little complicated now. And as a simple example, we first consider the mixed-state to pure-state case  $\xi: \{\rho_1, \rho_2, \dots, \rho_n\} \rightarrow \{|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle\}$ .

For any mixed input state  $\rho_i$ , it has a spectral decomposition

$$\rho_i = \sum_k |\tilde{\varphi}_k^{(i)}\rangle\langle\tilde{\varphi}_k^{(i)}|, \quad (9)$$

where the non-normalized state  $|\tilde{\varphi}_k^{(i)}\rangle = \sqrt{r_k^{(i)}}|\varphi_k^{(i)}\rangle$  ( $\| |\tilde{\varphi}_k^{(i)}\rangle \| = \sqrt{r_k^{(i)}}$ ) are orthogonal to each other (one can also choose other kinds of decompositions without affecting the generality of the problem we consider here). If  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j) = \emptyset$  for any  $i$  and  $j$ , we can obtain the following equations

$$U|\tilde{\varphi}_k^{(i)}\rangle|0\rangle = \sqrt{\eta_k^{(i)}}|\phi_i\rangle|\alpha_k^{(i)}\rangle|P_0\rangle + |\tilde{\beta}_k^{(i)}\rangle. \quad (10)$$

Here  $|\alpha_k^{(i)}\rangle$  are arbitrary states of subsystem  $a$ . If the intersection of the supports of the two density matrices  $\rho_i$  and  $\rho_j$  is not empty, there exists at least one state vector  $|\psi\rangle \in \text{supp}(\rho_i) \cap \text{supp}(\rho_j)$ . From the definition of the CP map, we have

$$\begin{aligned} U|\psi\rangle|0\rangle &= \sqrt{\eta^{(i)}}|\phi_i\rangle|\alpha_i\rangle|P_0\rangle + |\tilde{\beta}_i\rangle \\ &= \sqrt{\eta^{(j)}}|\phi_j\rangle|\alpha_j\rangle|P_0\rangle + |\tilde{\beta}_j\rangle. \end{aligned} \quad (11)$$

When the output states are *different* from each other, *i.e.*,  $|\phi_i\rangle \neq |\phi_j\rangle$ , the above equation becomes possible only when  $\eta_i = \eta_j = 0$ . Therefore any state with its support contained in  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j)$  has no contribution to

the desired transformation. Thus in this case, it is enough to consider  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j) = \emptyset$ . Otherwise, if  $|\phi_i\rangle = |\phi_j\rangle$ , the components in  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j)$  cannot be neglected.

Eq.(10) leads to

$$\tilde{X} - \sqrt{\Gamma}Y\sqrt{\Gamma}^\dagger \circ A \geq 0 \quad (12)$$

with

$$w = \begin{pmatrix} w_{ii} & \cdots & w_{ij} \\ \vdots & \ddots & \vdots \\ w_{ji} & \cdots & w_{jj} \end{pmatrix} \quad \{w \in (\tilde{X}, Y, A)\}, \quad (13)$$

where  $(\tilde{X}_{ij})_{kl} = \langle \tilde{\varphi}_k^{(i)} | \tilde{\varphi}_l^{(j)} \rangle$ ,  $(A_{ij})_{kl} = \langle \alpha_k^{(i)} | \alpha_l^{(j)} \rangle$ ,  $(Y_{ij})_{kl} = \langle \phi_i | \phi_j \rangle$ , and  $\Gamma = \text{diag}\{\Gamma_i, \dots, \Gamma_j\} = \text{diag}\{\text{diag}\{\eta_1^{(i)}, \eta_2^{(i)}, \dots\}, \dots, \text{diag}\{\eta_1^{(j)}, \eta_2^{(j)}, \dots\}\}$ .  $\tilde{X}_{ij}$  arises from the decomposition of input mixed states and  $A_{ij}$  is determined by the auxiliary subsystem  $a$ . Accordingly we arrive at the following lemma

**Lemma 3.** *For any given mixed state chosen from the set  $\{\sigma_i, \dots, \sigma_n\}$ , it can be transformed respectively to the corresponding pure output state  $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$  with the success probability matrix  $\Gamma$  if and only if Eq. (12) is satisfied. Moreover, if the input states are chosen with prior probabilities  $\{p_1, \dots, p_n\}$  with  $\sum_i p_i = 1$ , then the whole success probability will be  $P = \sum_i p_i \text{Tr}(\Gamma_i)$ .*

For any mixed state, it can be regarded as an ensemble of pure states. Eq. (12) means that the transformation from mixed input states to pure states is equivalent to transformation from pure states sets to pure states, i.e.  $\xi : \{\{|\tilde{\varphi}_1^{(i)}\rangle, |\tilde{\varphi}_2^{(i)}\rangle, \dots\}, \dots, \{|\tilde{\varphi}_1^{(j)}\rangle, |\tilde{\varphi}_2^{(j)}\rangle, \dots\}\} \rightarrow \{|\phi_i\rangle, \dots, |\phi_j\rangle\}$ , which makes the question a little different from the former cases. For example, we consider the deterministic transformation from mixed-state to pure-state. This indicates the efficiency matrix  $\Gamma_i = \text{diag}\{\sqrt{r_1^{(i)}}, \sqrt{r_2^{(i)}}, \dots\}$ . Now Eq. (12) can be written as

$$\begin{pmatrix} X_{ii} & \cdots & X_{ij} \\ \vdots & \ddots & \vdots \\ X_{ji} & \cdots & X_{jj} \end{pmatrix} = \begin{pmatrix} A_{ii} & \cdots & \langle \phi_i | \phi_j \rangle A_{ij} \\ \vdots & \ddots & \vdots \\ \langle \phi_j | \phi_i \rangle A_{ji} & \cdots & A_{jj} \end{pmatrix}$$

with  $(X_{ij})_{kl} = \langle \varphi_k^{(i)} | \varphi_l^{(j)} \rangle$ . Hence if one can find such a positive semidefinite matrix  $A$  (with its diagonal elements  $(A_{ii})_{kk} = 1$ ) to ensure that the above equation is satisfied, we can implement the desired transformation determinately. If  $\langle \phi_i | \phi_j \rangle \neq 0$ , one can construct the matrix  $A$  very easily  $A_{ij} = X_{ij} / \langle \phi_i | \phi_j \rangle$ . By judging the positive definiteness of  $A$ , we can get to know the existence of a deterministic transformation. As a simple case, we assume there are only two input states ( $\rho_1$  and  $\rho_2$ ) considered here. In the case of pure input states, deterministic transformation exists only when  $\langle \varphi_1 | \varphi_2 \rangle \leq F(\sigma_1, \sigma_2)$ . While in the mixed input states case, the above equation can be simplified into

$$\begin{pmatrix} I_{11} & X_{12}/\langle \phi_1 | \phi_2 \rangle \\ X_{21}/\langle \phi_2 | \phi_1 \rangle & I_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (14)$$

Here  $I_{11}$ ,  $I_{22}$  are both identity matrices. Thus  $A$  is positive semidefinite if and only if  $X_{12}X_{21} \leq |\langle \phi_1 | \phi_2 \rangle|^2 I_{11}$ . This means the maximal singular value of  $X_{12}$  which arises from the normalized eigenvectors of the two mixed input states must not be larger than the fidelity of the two output states  $|\langle \phi_1 | \phi_2 \rangle|$ .

#### D. CP trace non-increasing map between two mixed states sets

In the general case, the input and output states are both mixed. Since for a mixed state  $\rho_i = \sum_k |\tilde{\varphi}_k^{(i)}\rangle \langle \tilde{\varphi}_k^{(i)}| = \sum_m |\tilde{\psi}_m^{(i)}\rangle \langle \tilde{\psi}_m^{(i)}|$ , it can be generated from very many different kinds of ensembles. This will make the question more complicated. Following the same routine as before, we can write down the unitary implementation of the CP map

$$U|\tilde{\varphi}_k^{(i)}\rangle|0\rangle = |\tilde{\phi}_k^{(i)}\rangle|P_0\rangle + |\tilde{\beta}_k^{(i)}\rangle, \quad (15)$$

and the output states  $|\tilde{\phi}_k^{(i)}\rangle$  should satisfy  $\sum_k \text{Tr}_a(|\tilde{\phi}_k^{(i)}\rangle \langle \tilde{\phi}_k^{(i)}|) = \eta_i \sigma_i$ , where  $\eta_i$  describes the success probability of the CP map on  $\rho_i$ . When the output or the input states are pure, one can check that  $|\tilde{\phi}_k^{(i)}\rangle$  are proportional to the purifications of the output states. Also if the output states are different from each other and their supports have no common nonzero vectors, we can obtain the states lying in  $\text{supp}(\rho_i) \cap \text{supp}(\rho_j)$  have no contribution the transformation.

Similarly, the inner-product preservation of unitary transformation lead us to

$$\begin{pmatrix} \tilde{X}_{ii} & \cdots & \tilde{X}_{ij} \\ \vdots & \ddots & \vdots \\ \tilde{X}_{ji} & \cdots & \tilde{X}_{jj} \end{pmatrix} - \begin{pmatrix} \tilde{Y}_{ii} & \cdots & \tilde{Y}_{ij} \\ \vdots & \ddots & \vdots \\ \tilde{Y}_{ji} & \cdots & \tilde{Y}_{jj} \end{pmatrix} \geq 0, \quad (16)$$

where  $(\tilde{Y}_{ij})_{kl} = \langle \tilde{\phi}_k^{(i)} | \tilde{\phi}_l^{(j)} \rangle$ . Therefore we have

**Lemma 4.** *The transformation  $\xi$  from mixed states to mixed state  $\xi : \{\rho_i, \dots, \rho_j, \dots\} \rightarrow \{\sigma_i, \dots, \sigma_j, \dots\}$  can be implemented with the success probabilities  $\{\eta_i, \dots, \eta_j, \dots\}$  if and only if there exists sets of composite states  $\theta_i = |\tilde{\phi}_k^{(i)}\rangle \langle \tilde{\phi}_k^{(i)}|$  with  $\sum_k \text{Tr}_a(\theta_i) = \eta_i \sigma_i$  such that Eq. (16) is satisfied. If the input states are chosen with prior probability  $\{p_i, \dots, p_j, \dots\}$  and  $\sum_i p_i = 1$ , then the whole success probability will be  $P = \sum_i p_i \eta_i = \sum_i p_i \text{Tr}(\tilde{Y}_{ii})$ .*

Lemma 4 characterizes the most general properties of CP maps between mixed states. By assuming the output or the input states are pure, we can immediately obtain the corresponding results in section (II A – II C). To make our result more specific, let us focus on an interesting case where the output states  $\{\sigma_i\}$  are all orthogonal to each other. This question is a very special case of transformation between mixed states and has

drawn much attention recently (unambiguous discrimination of mixed states) [14]. Since  $\sigma_i \perp \sigma_j$ , we have  $\langle \tilde{\phi}_k^{(i)} | \tilde{\phi}_l^{(j)} \rangle = 0$  ( $i \neq j$ ) for any  $k$  and  $l$ . This implies that the matrix  $\tilde{Y}$  is quasi-diagonal and can be expressed as  $\tilde{Y} = \text{diag}\{\tilde{Y}_{ii}, \dots, \tilde{Y}_{jj}, \dots\}$ . Hence we obtain that  $N$  mixed states  $\{\rho_1, \rho_2, \dots\}$  can be unambiguously discriminated if and only if there exists a positive semidefinite quasi-diagonal matrix  $\tilde{Y} = \text{diag}\{\tilde{Y}_{11}, \tilde{Y}_{22}, \dots\}$  such that  $\tilde{X} - \tilde{Y} \geq 0$ .

As another interesting example, in the following we assume there are only two input states contained here. In this case, Eq. (16) can be rewritten as

$$\begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix} - \begin{pmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{pmatrix} \geq 0. \quad (17)$$

Without loss of generality, we also suppose both  $\tilde{X}_{11}$  and  $\tilde{X}_{22}$  are  $t \times t$  matrices. From the standard linear algebra theory, we obtain the following inequalities

$$\sqrt{(r_k^{(1)} - \eta_k^{(1)})(r_k^{(2)} - \eta_k^{(2)})} \geq |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle| - \langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle. \quad (18)$$

It's not difficult to find that

$$p_1 \eta_1 + p_2 \eta_2 \leq \frac{1 - 2\sqrt{p_1 p_2} \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle|}{1 - \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle| / \sqrt{\eta_1 \eta_2}}. \quad (19)$$

Since different ensembles can give rise to the same mixed states, this is known as the unitary freedom for density matrices, we have

$$P \leq \frac{1 - 2\sqrt{p_1 p_2} \max_{\{|\tilde{\phi}_k^{(1)}\rangle, |\tilde{\phi}_k^{(2)}\rangle\}} \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle|}{1 - \max_{\{|\tilde{\phi}_k^{(1)}\rangle, |\tilde{\phi}_k^{(2)}\rangle\}} \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle| / \sqrt{\eta_1 \eta_2}}, \quad (20)$$

where  $P = p_1 \eta_1 + p_2 \eta_2$ , and the maximization is over all decompositions of the input mixed states satisfying  $\rho_i = \sum_k |\tilde{\phi}_k^{(i)}\rangle \langle \tilde{\phi}_k^{(i)}|$ . Interestingly, the right hand side of Eq. (20) is directly related to the fidelity of the two input mixed states  $F(\rho_1, \rho_2) = \text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}$ . To show this, we introduce the purification of  $\rho_1$  and  $\rho_2$

$$\begin{aligned} |\varphi_1\rangle &= \sqrt{\rho_1} U^{(1)} \otimes U_Q^{(1)} |m\rangle, \\ |\varphi_2\rangle &= \sqrt{\rho_2} U^{(2)} \otimes U_Q^{(2)} |m\rangle, \end{aligned}$$

where  $|m\rangle = \sum_i |i\rangle |i_Q\rangle$  is, up to a normalization factor, a maximally entangled state of the input system and the extended system  $Q$  (we assume that the two systems have the same rank) [12]. Choosing  $U_Q^{(1)} = U_Q^{(2)}$ , one can find suitable unitary matrices  $U^{(1)}$  and  $U^{(2)}$  such that

$$\langle \varphi_1 | \varphi_2 \rangle = \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle|. \quad (21)$$

From the Uhlmann's theorem [12, 13] we have

$$\begin{aligned} F(\rho_1, \rho_2) &= \max_{\{|\varphi_1\rangle, |\varphi_2\rangle\}} \langle \varphi_1 | \varphi_2 \rangle \\ &= \max_{\{|\tilde{\phi}_k^{(1)}\rangle, |\tilde{\phi}_k^{(2)}\rangle\}} \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle|. \end{aligned}$$

Thus Eq. (20) now can be simplified as

$$\begin{aligned} P &\leq \frac{1 - 2\sqrt{p_1 p_2} F(\rho_1, \rho_2)}{1 - \max_{\{|\tilde{\phi}_k^{(1)}\rangle, |\tilde{\phi}_k^{(2)}\rangle\}} \sum_k |\langle \tilde{\phi}_k^{(1)} | \tilde{\phi}_k^{(2)} \rangle| / \sqrt{\eta_1 \eta_2}} \\ &\leq \frac{1 - 2\sqrt{p_1 p_2} F(\rho_1, \rho_2)}{1 - F(\sigma_1, \sigma_2)}. \end{aligned} \quad (22)$$

This result is actually the generalize version of Eq. (8). One can check that when  $\sigma_1 \perp \sigma_2$ ,  $P \leq 1 - 2\sqrt{p_1 p_2} F(\rho_1, \rho_2)$ , which is also consistent with already known results on unambiguous discrimination between two mixed states [14].

### III. THE ROLE OF AUXILIARY SYSTEM AND SEMIDEFINITE PROGRAMMING

In the above sections, we have considered the condition under which a physical accessible transformation exists on a finite number of input states. By constructing the unitary realization of the CP maps, we have presented the sufficient and necessary conditions on the existence of the desired transformation. The auxiliary system  $a$  plays a very important role in our consideration. Usually in order to implement a CP transformation it is necessary to introduce such a subsystem. For instance, in the deterministic transformation between pure states, if  $|\langle \varphi_1 | \varphi_2 \rangle| < |\langle \phi_1 | \phi_2 \rangle|$ , it is impossible to complete the transformation  $\{|\varphi_1\rangle, |\varphi_2\rangle\} \rightarrow \{|\phi_1\rangle, |\phi_2\rangle\}$  without the ancilla system  $a$ .

As we have mentioned before, probabilistic cloning can be regarded as a special transformation on a finite number of linear independent states. However, in the original work [6], the subsystem  $a$  has been neglected, and the corresponding unitary map is defined as

$$U|\varphi_i\rangle|0\rangle = \sqrt{\eta_i}|\varphi_i\rangle^{\otimes N}|P_0\rangle + |\tilde{\beta}_i\rangle. \quad (23)$$

This is equivalent to saying that probabilistic cloning can be implemented by setting all  $|\alpha_i\rangle$  in Eq. (1) to be equal, or equivalently  $A$  is a rank-1 operator. Generally it is difficult to say this since  $X - \sqrt{\Gamma}Y\sqrt{\Gamma}^\dagger \circ A \geq 0$  can not ensure  $X - \sqrt{\Gamma}Y\sqrt{\Gamma}^\dagger \geq 0$ . This also indicates that in some cases we can find a probability matrix  $\Gamma$  which satisfies  $X - \sqrt{\Gamma}Y\sqrt{\Gamma}^\dagger \not\geq 0$ , i.e., it cannot be implemented by the probabilistic copy machine in [6], but can be realized in our framework if one can find a suitable matrix  $A$ . In this sense, we define the generalized probabilistic cloning machine as

$$U|\varphi_i\rangle|0\rangle = \sqrt{\eta_i}|\varphi_i\rangle^{\otimes N}|\alpha_i\rangle|P_0\rangle + |\tilde{\beta}_i\rangle. \quad (24)$$

However, if the input set contains only two states, the optimal probabilistic copy machine can be implemented without the ancilla system  $a$  [6].

In a more realistic situation, people often concentrate on the whole success probability of such physical transformation. This indicates that we should make the prob-

ability  $P$  as high as possible. Interestingly, if the subsystem  $a$  is contained and if we know exactly the two positive semidefinite matrices  $X$  and  $Y$  (this happens when the output states are pure or orthogonal to each other), we can reduce this to a standard semidefinite program (SDP) problem [15].

Usually a standard SDP problem is to maximize

$$\sum_m b_m y_m \quad (25)$$

subject to

$$C - \sum_{i=1}^n y_m S_m \geq 0, \quad (26)$$

where  $b \in R^n$ ,  $y \in R^n$  with  $b$  be a given vector,  $b_m, y_m$  are the  $m$ th components of vector  $b$  and  $y$  respectively, and  $C$  and  $S_m$  are given Hermitian matrices. While in our framework we should maximize (for example, see Lemma 1)

$$P = \sum_k p_k \tilde{A}_{kk} \quad (27)$$

under the constraints

$$X - Y \circ \tilde{A} \geq 0, \text{ and } \tilde{A} \geq 0. \quad (28)$$

Here we have assumed  $\tilde{A} = \sqrt{\Gamma} A \sqrt{\Gamma}^\dagger$  for simplicity. Now we define

$$\begin{aligned} F_0 &= \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \\ F_{kl} &= \begin{pmatrix} Y_{kl} E_{kl} + Y_{lk} E_{lk} & 0 \\ 0 & -E_{kl} - E_{lk} \end{pmatrix}, \\ G_{kl} &= i \begin{pmatrix} Y_{kl} E_{kl} - Y_{lk} E_{lk} & 0 \\ 0 & -E_{kl} + E_{lk} \end{pmatrix}, \end{aligned}$$

where  $i$  is the basic imaginary unit with  $i^2 = -1$ ,  $Y_{kl}$  denotes the  $(k, l)$ th entry of the matrix  $Y$ , and  $E_{kl}$  are matrices with all entries be zero except the  $(k, l)$ th elements  $(E_{kl})_{mn} = \delta_{km} \delta_{ln}$ . Eq. (27) can be reformulated as

$$\max_{\tilde{A}} \sum_k p_k \tilde{A}_{kk} \quad (29)$$

subject to

$$\begin{aligned} F_0 - \sum_{k,l>k} \left[ \text{Re}(\tilde{A}_{kl}) F_{kl} + \text{Im}(\tilde{A}_{kl}) G_{kl} \right] \\ - \frac{1}{2} \sum_k \tilde{A}_{kk} F_{kk} \geq 0, \end{aligned} \quad (30)$$

where  $\text{Re}(\tilde{A}_{kl})$  and  $\text{Im}(\tilde{A}_{kl})$  represent the real and imaginary part of  $\tilde{A}_{kl}$  respectively. Thus, this problem has been reduced to a standard SDP problem, which can be solved numerically to  $\epsilon$ -optimization in polynomial time.

It can be found that the subsystem  $a$  plays a very important role in the reducing step, which ensures the elements  $\tilde{A}_{kl}$  to be independent of each other. Otherwise, we cannot reduce the whole problem to SDP easily. Moreover, to judge the existence of the matrix  $\tilde{A}$  now is equivalent to determining where there exist  $\tilde{A}_{kl}$  to make Eq. (30) satisfied. This is often called semidefinite feasibility problem (SDFP). Unfortunately, this problem has not been totally solved yet [15].

As a specific example, consider the case where the input contains three states with equal prior probability, i.e.,  $|\varphi_1\rangle = (2|0\rangle + |1\rangle + |2\rangle)/\sqrt{6}$ ,  $|\varphi_2\rangle = (|0\rangle + 3|1\rangle + |2\rangle)/\sqrt{11}$ , and  $|\varphi_3\rangle = (|0\rangle + |1\rangle + 4|2\rangle)/(3\sqrt{2})$ . We concentrate on a general state separation problem (generalized version of probabilistic cloning), that is to transform these input states into the following output states  $|\phi_1\rangle = (10|0\rangle + |1\rangle + |2\rangle)/\sqrt{101}$ ,  $|\phi_2\rangle = (|0\rangle + 10|1\rangle + |2\rangle)/\sqrt{101}$ , and  $|\phi_3\rangle = (|0\rangle + |1\rangle + 10|2\rangle)/\sqrt{101}$  separately. The optimal solution can be found when

$$A = \begin{pmatrix} 0.1570 & 0.2329 & 0.2643 \\ 0.2329 & 0.4342 & 0.2977 \\ 0.2643 & 0.2977 & 0.5452 \end{pmatrix}. \quad (31)$$

This matrix has two non-zero eigenvalues with  $\lambda_1 = 0.0$ ,  $\lambda_2 = 0.1874$ , and  $\lambda_3 = 0.9491$ . The corresponding success probability matrix is  $\Gamma = \text{diag}\{0.1570, 0.4342, 0.5453\}$ . We can also check that  $X - \sqrt{\Gamma} Y \sqrt{\Gamma}$  is not a positive semidefinite matrix, hence the corresponding transformation cannot be implemented without the subsystem  $a$ , which agrees with the above presentation.

#### IV. CONCLUSION

In summary, we have considered physical accessible transformations on a finite set of input states. From the system-ancilla model, by constructing the appropriate unitary map, we obtain the sufficient and necessary conditions on the existence of the desired transformation. Many interesting questions can be enclosed in this framework, such as deterministic transformation between quantum states, probabilistic cloning of linear independent states, state separation, unambiguous state discrimination, *etc.* Our discussion reveals that all these problems can be treated uniformly, and actually they are only special cases of general CP maps on a finite input states set. In the practical viewpoint, usually people have only a finite number of different states in hand, therefore to explore what kinds of operations we can do and how to do on these quantum resources becomes a very important problem. We expect our results will be useful in judging the existence of the CP maps and also helpful in constructing these transformations.

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